MATHEMATICS

AN ENVELOPING SERIES FOR THE ZETA FUNCTION

BY

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The concept of an enveloping series was introduced by G. PÓLYA and G. SZEGÖ [1]. A convergent or divergent series of terms a_n is said to envelop a number A if for each $n=0, 1, \ldots$

(1) $$A-(a_0+a_1+\ldots+a_n)=\theta_n a_{n+1} \text{ with } 0<\theta_n<1.$$

In this case we write

(2) $$A \propto \sum_{n=0}^{\infty} a_n.$$

A similar notion had been developed by G. A. SCHOTT and G. N. WATSON [2]. The concept has been generalized and applied extensively by J. G. VAN DER CORPUT [3].

The purpose of this paper is to obtain an enveloping series for the function

(3) $$\zeta(s)-\sum_{n=1}^{a-1} n^{-s}=\zeta(s,a),$$

where a is a positive integer and $\zeta(s)$ is the zeta function of RIEMANN. The function $\zeta(s, a)$ is the zeta function of HURWITZ [4], p. 265.

Let $\operatorname{Re} s=\sigma$. We have [4], p. 266,

(4) $$\zeta(s,a)\,\Gamma(s)=\int_0^{\infty} \frac{x^{s-1}e^{-ax}}{1-e^{-x}}\,dx \qquad (\sigma>1).$$

Using the identity

(5) $$(1-e^{-x})^{-1}=\frac{1}{x}+\frac{1}{2}+\left(-\frac{1}{x}+\frac{1}{2}\coth\frac{x}{2}\right),$$

we find

(6) $$\zeta(s,a)\,\Gamma(s)=\frac{\Gamma(s-1)}{a^{s-1}}+\frac{1}{2}\frac{\Gamma(s)}{a^s}+\int_0^{\infty} e^{-ax}x^{s-1}\left(-\frac{1}{x}+\frac{1}{2}\coth\frac{x}{2}\right)dx.$$

By analytic continuation, this identity holds for $\sigma>-1$.

We have [4], p. 126,

(7) $$-\frac{1}{x}+\frac{1}{2}\coth\frac{x}{2}=\int_0^{\infty}\frac{\sin\dfrac{xt}{2}}{e^{\pi t}-1}\,dt.$$

But the sine function is enveloped by its MacLaurin series [1], p. 26. Since $e^{\pi t}-1>0$ for $t>0$, we may use (7) to obtain the result

$$(8)\qquad -\frac{1}{x}+\frac{1}{2}\coth\frac{x}{2} \propto \sum_{k=1}^{\infty}\frac{(-)^{k-1}B_k}{(2k)!}x^{2k-1}\qquad(-\infty<x<\infty),$$

where the Bernoulli numbers B_k are given by the integrals [4], p. 126,

$$(9)\qquad B_k=2^{2-2k}\int_0^{\infty}\frac{t^{2k-1}\,dt}{e^{\pi t}-1}.$$

For real $s>-1$, we may use the enveloping series (8) in an integration to obtain

$$(10)\qquad \int_0^{\infty}e^{-ax}x^{s-1}\left(-\frac{1}{x}+\frac{1}{2}\coth\frac{x}{2}\right)dx \propto \sum_{k=1}^{\infty}\frac{(-)^{k-1}B_k\,\Gamma(s+2k-1)}{(2k)!\,a^{s+2k-1}}.$$

From (6) we now find the enveloping series

$$(11)\qquad \zeta(s,a)-\frac{1}{(s-1)\,a^{s-1}}-\frac{1}{2a^s} \propto \sum_{k=1}^{\infty}\frac{(-)^{k-1}B_k\,s(s+1)\dots(s+2k-2)}{(2k)!\,a^{s+2k-1}}.$$

The result (11), which holds for real $s>-1$ may be extended to real values $\leqslant -1$ and to complex values. Let

$$(12)\qquad R_n(x)=-\frac{1}{x}+\frac{1}{2}\coth\frac{x}{2}-\sum_{k=1}^{n}\frac{(-)^{k-1}B_k}{(2k)!}x^{2k-1}.$$

The remainder $R_n(x)$ is $O(x^{2n+1})$ at the origin. Therefore, by analytic continuation, the identity

$$(13)\qquad \left\{\begin{aligned}\int_0^{\infty}e^{-ax}x^{s-1}R_n(x)\,dx=\zeta(s,a)\,\Gamma(s)-\frac{\Gamma(s-1)}{a^{s-1}}-\frac{1}{2}\frac{\Gamma(s)}{a^s}-&\\ -\sum_{k=1}^{n}\frac{(-)^{k-1}B_k\,\Gamma(s+2k-1)}{(2k)!\,a^{s+2k-1}}&\end{aligned}\right.$$

holds for all complex s with $\sigma>-2n-1$. But, by (8) and (12),

$$(14)\qquad R_n(x) \propto \sum_{k=n+1}^{\infty}\frac{(-)^{k-1}B_k}{(2k)!}x^{2k-1}\qquad(-\infty<x<\infty).$$

Using this enveloping series in (13), we find that for all real $s>-2n-1$

$$(15)\qquad \zeta(s,a)=\frac{1}{(s-1)\,a^{s-1}}+\frac{1}{2a^s}+\sum_{k=1}^{n}\frac{(-)^k B_k\,s(s+1)\dots(s+2k-2)}{(2k)!\,a^{s+2k-1}}+T_n,$$

where T_n is the integral (13) divided by $\Gamma(s)$. By (14)

$$(16)\qquad T_n \propto \sum_{k=n+1}^{\infty}\frac{(-)^k B_k\,s(s+1)\dots(s+2k-2)}{(2k)!\,a^{s+2k-1}}.$$

Finally suppose that s is complex, with $\sigma > -2n-1$. In this case the identity (15) still holds, but the enveloping relation (16) fails. Using the enveloping series (14) for $R_n(x)$ in the integral (13), we find for each $N \geqslant n+1$

$$(17) \quad T_n = \sum_{k=n+1}^{N} \frac{(-)^k B_k s(s+1)\dots(s+2k-2)}{(2k)!\, a^{s+2k-1}} + \psi_N \frac{B_{N+1}\, \Gamma(\sigma+2N+2)}{(2N+2)!\, \Gamma(s)\, a^{\sigma+2N+1}},$$

where ψ_N is a complex number with $|\psi_N| < 1$. If s is a complex number with $\sigma > -2n-1$, the result (15) may be used where the relation (17) is substituted for the strict enveloping relation (16).

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